

# Rigel DG Lie Algebras

Talk 6

① Reminder

$K$  — field of characteristic zero

② Let  $\text{Vect}_K^{dg}$  denote the category of dg vector spaces

$$(\dots \rightarrow V_1 \xrightarrow{d} V_0 \xrightarrow{d} V_{-1} \rightarrow \dots) \quad \text{s.t. } d^2 = 0$$

and maps are chain morphisms

③ Let  $\text{Alg}_K^{dg}$  denote the category of dg associative algebras:

$$A_+ = (\dots \rightarrow A_1 \xrightarrow{d} A_0 \xrightarrow{d} A_{-1} \rightarrow \dots)$$

&  $d$  is a derivation for multiplication

④ Let  $\text{CAlg}_K^{dg}$  denote the category of dg commutative algebras:

$$A_+ \in \text{CAlg}_K^{dg} \quad \text{s.t.} \quad \forall x, y, \quad xy = (-1)^{|y|x} yx$$

⑤ Let  $\text{Lie}_K^{dg}$  denote the category of dg Lie algebras:

$$(g_+, d) \in \text{Vect}_K^{dg} \quad \& \quad [ , ] : g_+ \times g_+ \rightarrow g_+$$

deg 0, ~~skew-symmetric~~, bilinear s.t.

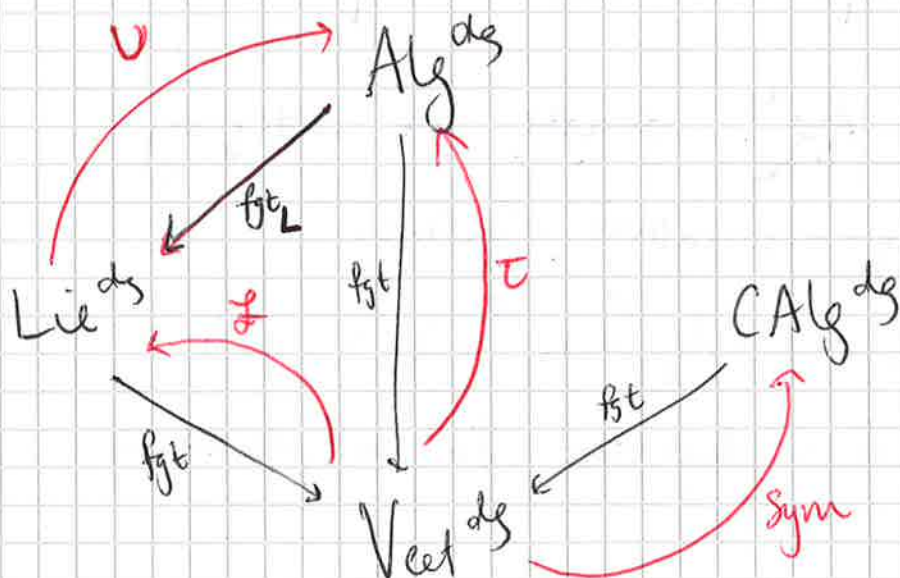
$$1) [x, y] = -(-1)^{xy} [y, x] \quad (\text{skew-symmetry})$$

$$2) [x [y, z]] = [[x, y] z] + (-1)^{xy} [y [x, z]] \quad (\text{Jacobi})$$

3)  $d$  is a derivation of  $[\cdot, \cdot]$

## Examples of functors

How do these relate?



$f_{gt} = \text{"forget"}$

There is a "not so forgetful" functor

$$\text{Alg}^{dg} \xrightarrow{f_{gt\_L}} \text{Lie}^{dg}$$

$$(A, \cdot) \longrightarrow (A, [\cdot])$$

$$[a, b] = a \cdot b - (-1)^{ab} b \cdot a$$

tensor algebra functor  
There is  $\mathbb{T}$ :  $\text{Vect}^{\text{dg}} \rightarrow \text{Alg}^{\text{dg}}$

$$V_* \longmapsto \mathbb{T}(V_*) = \bigoplus_n (V_*)^{\otimes n}$$

multiplication = concatenation

There is the free Lie algebra functor

$$\mathbb{F}: \text{Vect}^{\text{dg}} \rightarrow \text{Lie}^{\text{dg}}$$

where  $\mathbb{F}(V_*)$  is the sub dg Lie algebra in

$\text{fgt}_L(\mathbb{T}(V_*))$  generated by  $V_* \cong V_*^{\otimes 1} \hookrightarrow \mathbb{T}(V_*)$ .

There is the universal enveloping algebra functor

$$\mathbb{U}: \text{Lie}^{\text{dg}} \rightarrow \text{Alg}^{\text{dg}}$$

where

$$\mathbb{U}(\mathfrak{g}_*) = \mathbb{T}(\mathfrak{g}_*) / \mathcal{I}_{\mathfrak{g}}$$

with  $\mathcal{I}_{\mathfrak{g}}$  is the two-sided ideal generated

$$\text{by } \{ x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] : x, y \in \mathfrak{g} \}.$$

The objects  $\mathbb{F}(V_*)$ ,  $\mathbb{T}(V_*)$ ,  $\mathbb{U}(\mathfrak{g}_*)$  satisfy  
universal properties (as they come  
from left adjoints)

eg

Ex

$$\mathfrak{g}_* \in \text{Lie}^{\text{ds}}, \quad A \in \text{Alg}^{\text{ds}}$$

given  $\varphi: \mathfrak{g}_* \rightarrow \text{fgt}_L(A_*)$  a map of dg Lie algs,  
there is a unique map of assoc dg algs

$$\tilde{\varphi}: U\mathfrak{g}_* \rightarrow A_*$$

such that

$$\begin{array}{ccc} \mathfrak{g}_* & \xrightarrow{\varphi} & A_* \\ \downarrow U & \circlearrowleft & \downarrow \tilde{\varphi} \\ U\mathfrak{g}_* & & A_* \end{array}$$

In short, we have

$$\tau \dashv \text{fgt}, \quad \mathbb{F} \dashv \text{fgt}, \quad \mathcal{U} \dashv \text{fgt}_L$$

left adjoints

More about  $U(\mathfrak{g}_*)$

There is a natural filtration

$$T(\mathfrak{g}) : \mathfrak{g}^{\otimes 0} \subseteq \mathfrak{g}^{\otimes 0} \oplus \mathfrak{g}^{\otimes 1} \subseteq \dots \subseteq \bigoplus_{i=0}^n \mathfrak{g}^{\otimes i} \subseteq \dots$$

$\downarrow$

$$U(\mathfrak{g}) \quad k \subseteq \dots \subseteq (U\mathfrak{g})^{\leq n} \subseteq \dots$$

$$\text{Let } \text{gr}(U\mathfrak{g}) = \bigoplus_{n=0}^{\infty} (U\mathfrak{g})^{\leq n} / (U\mathfrak{g})^{\leq n-1}$$

It is a commutative dg algebra.

Then composite

$$\mathfrak{g} \hookrightarrow U^{\leq 1}(\mathfrak{g}) \longrightarrow \text{gr}(U\mathfrak{g})$$

induces a map of dg commutative algebras

$$\psi_{\mathfrak{g}}: \text{Sym}(\mathfrak{g}) \longrightarrow \text{gr}(U\mathfrak{g})$$

Thm (Poincaré-Birkhoff-Witt)

$\psi_{\mathfrak{g}}$  is an isomorphism.

Pf Tedious: pick a basis.  $\square$

② Upgrade to Quillen adjunctions

Equip  $\text{Vect}^{\text{dg}}$  with a model category structure where

$$W = \{ \text{quasi-isomorphisms} \}$$

$$F = \{ f_n: M_n \rightarrow N_n : f_n \text{ surjective } \forall n \}$$

$$\text{Cof} = \{ \text{degree-wise surjective} \}$$

We can transfer this model structure to  $\text{Lie}^{\text{ds}}$ ,  $\text{Alg}^{\text{ds}}$ , and  $\text{CATg}^{\text{ds}}$  via our adjunctions  $(\mathbb{F}, \mathbb{T}, \text{and } \text{Sym}, \text{respectively})$ .

For example,

$$W_{\text{Lie}} = W \cap \text{Mor}_{\text{Lie}} = \text{fgt}^{-1}(W)$$

$$F_{\text{Lie}} = F \cap \text{Mor}_{\text{Lie}} = \text{fgt}^{-1}(F)$$

This generates a cofibrantly-generated model category on  $\text{Lie}^{\text{ds}}$ .

Remark: On  $\text{CATg}^{\text{ds}}$ , the analogous structure lets you begin DAG in characteristic zero. //

Thus  $(\mathbb{F}, \text{fgt})$ ,  $(\mathbb{T}, \text{fgt})$ ,  $(\text{Sym}, \text{fgt})$  are Quillen adjunctions!

In fact,  $(U, \text{fgt}_L)$  is a Quillen adjunction too.

~~Lemma~~

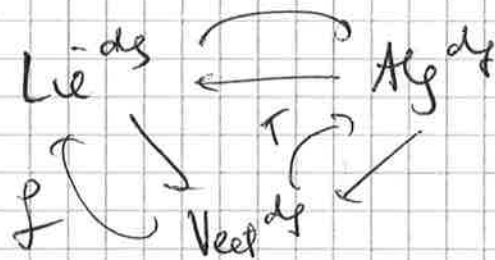
## Lemma

If  $\phi: \mathfrak{g}_* \rightarrow \mathfrak{g}'_*$  is a  $\mathbb{Q}$ -Lie algebra map,  
then  $\phi$  is a quasi-iso  $\Leftrightarrow U\phi$  is a quasi-iso.

## Proof

$\phi$  is a quasi-iso  $\stackrel{\text{char zero}}{\Leftrightarrow} \text{Sym}(\phi)$  is a quasi-iso  
 $\stackrel{\text{PBW}}{\Leftrightarrow} U(\phi)$  is a quasi-iso  $\square$

Thus we have a triangle of Quillen adjunctions



## ③ Lie algebra (co)homology

$\mathbb{A}$  is equivalent to equip  $M_*$  with  
a left  $U(\mathfrak{g}_*)$ -module structure

and a left  $\mathfrak{g}_*$ -representation structure.

Reminder:

$$\text{invariant elements} = M^{\mathfrak{g}} = \{m \in M : \mathfrak{g} \cdot m = 0\}$$

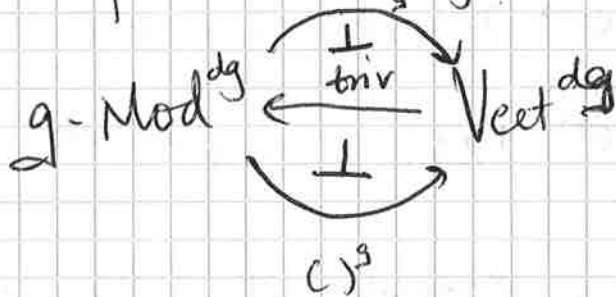
$$\text{coinvariants} = M_{\mathfrak{g}} = M / \mathfrak{g}M$$

Moreover

$$M^{\mathfrak{g}} = \text{Hom}_{\mathfrak{g}}(k, M)$$

$$M_{\mathfrak{g}} = k \otimes_{U_{\mathfrak{g}}} M$$

These are parts of adjunctions



triv(M)  
" trivial  $\mathfrak{g}$ -module structure

Hence  $(-)^{\mathfrak{g}} = \text{left exact}$

$(-)_{\mathfrak{g}} = \text{right exact}$

Def The Lie algebra homology of  $\mathfrak{g}$  is the left derived functor

$$L_*(-)_{\mathfrak{g}} =: H_*(\mathfrak{g}, -)$$

and the Lie algebra cohomology of  $\mathfrak{g}$  is the right derived functor

$$R_*(-)_{\mathfrak{g}} =: H^*(\mathfrak{g}, -)$$



To calculate (co)homology, find a projective resolution of  $k$  as a  $U\mathfrak{g}$ -module.

Idea: find an acyclic dg La containing  $\mathfrak{g}_k$   
then take  $\cup$  of it to obtain  
a resolution of  $k$

Construction

Define the "cone" of  $\mathfrak{g}$ ,  $C_n(\mathfrak{g})_*$  by

$$C_n(\mathfrak{g})_n = \mathfrak{g}_n \oplus \overset{\text{formal notation}}{\mathfrak{g}_{n-1}}$$

with differential

$$D(x + \varepsilon y) = dx + y - \varepsilon dy$$

*diff in  $\mathfrak{g}$*

and bracket

$$[x + \varepsilon y, x' + \varepsilon y'] = [x, x'] + \varepsilon \overset{(-1)^x}{[x, y']} + [y, x']$$

Remark  $C_n(\mathfrak{g})_*$  is acyclic &

$$\mathfrak{g} \xrightarrow{d_*} C_n(\mathfrak{g})_*$$

$$x \longmapsto x + \varepsilon 0$$

is an inclusion

Def The Chervalley-Eilenberg complex is

$$C_*(\mathfrak{g}, M) := U(\mathfrak{Cn}(\mathfrak{g})_+) \underset{U\mathfrak{g}}{\otimes} M$$

Rule  $\mathfrak{e}\mathfrak{g}[\mathfrak{l}] \hookrightarrow \mathfrak{Cn}(\mathfrak{g})_+$  is an abelian subalgebra so  $U(\mathfrak{e}\mathfrak{g}[\mathfrak{l}]) = \text{Sym}(\mathfrak{g}[\mathfrak{l}])$

By PBW, ~~the underlying vector space~~ <sup>graded</sup>

$$U(\mathfrak{Cn}(\mathfrak{g})_+) \underset{\cong}{\cong} \text{Sym}(\mathfrak{g}[\mathfrak{l}]) \underset{\otimes}{\otimes} U\mathfrak{g}$$

as  $U\mathfrak{g}$ -modules

$$\Rightarrow C_*(\mathfrak{g}) \underset{\cong}{\cong} \text{Sym}(\mathfrak{g}[\mathfrak{l}])$$

"  $\swarrow$  as graded vector spaces

$$C_*(\mathfrak{g}, k)$$

What is the differential?

It is given by

$$d(x_1, \dots, x_n) = \sum_i (-1)^{x_1 \dots x_{i-1}} (dx_i) x_{i+1} \dots x_n$$

$$+ \sum (-1)^{\sim} x_1 \dots x_{i-1} x_{i+1} \dots x_j [x_i, x_j] x_{j+1} \dots x_n$$

$C_*(\mathfrak{g})$  has a natural filtration by

$$C_*(\mathfrak{g})^{\leq n} = \bigoplus_{i=0}^n \text{Sym}^i(\mathfrak{g}(1))$$

so

$$\text{gr } C_*(\mathfrak{g}) \cong \text{Sym}(\mathfrak{g}_1)$$

as dg vector spaces

By definition,

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$$

Moreover

$$H^1(\mathfrak{g}, M) = \text{Der}(\mathfrak{g}, M)$$

$\mathbb{K}$  is a "universal" Lie module:

every Lie algebra has the trivial rep

$$\Rightarrow \mathfrak{g}_* \xrightarrow{\varphi} \mathfrak{g}'_* \rightsquigarrow C_*(\mathfrak{g}_*) \xrightarrow{C_*(\varphi)} C_*(\mathfrak{g}'_*)$$

That is, we have

$$C_* : \text{Lie}^{\text{dg}} \longrightarrow \text{coAlg}^{\text{dg}}$$

$$\text{Take } C^* \mathfrak{g} := (C_* \mathfrak{g})^\vee = \text{Hom}_k(C_* \mathfrak{g}, k)$$

Then  $C^* : \text{Lie}^{\text{dg}} \rightarrow (\text{coAlg}^{\text{ts}})^{\text{op}}$

DAG X, 2.2.6

Both  $C_+$ ,  $C^*$  preserve quasi-isos &

hence have adjoints at the level  
of homotopy categories

DAG X, 2.3.2

They do not admit right adjoints &

hence no Quillen adjoints

