Rigel DG Lie Algebras Talk 6

① Reminder

k — field of characteristic zero

ⓐ Let Vect_k^{dg} denote the category of dg vector spaces

$$(\cdots \to V_1 \xrightarrow{d} V_0 \xrightarrow{d} V_{-1} \to \cdots) \quad w/ \ d^2 = 0$$

and maps are chain morphisms

ⓑ Let Alg_k^{dg} denote the category of dg associative algebras:

$$A_* = (\cdots \to A_1 \xrightarrow{d} A_0 \xrightarrow{d} A_{-1} \to \cdots)$$

& d is a derivation for multiplication

ⓒ Let CAlg_k^{dg} denote the category of dg commutative algebras:

$A_* \in \mathrm{Alg}_k^{dg}$ s.t. $\forall x, y$, $xy = (-1)^{yx} yx$

ⓓ Let Lie_k^{dg} denote the category of dg Lie algebras:

$$(\mathfrak{g}_*, d) \in \mathrm{Vect}_k^{dg} \quad \& \quad [\,,\,] : \mathfrak{g}_* \times \mathfrak{g}_* \to \mathfrak{g}_*$$

deg 0, ~~skew symmetric~~ & bilinear s.t.

1) $[x,y] = -(-1)^{xy}[y,x]$ [skew-symmetry]

2) $[x[y\,z]] = [[x\,y]z] + (-1)^{xy}[y[x\,z]]$ [Jacobi]

3) d is a derivation of $[\,,\,]$

Examples of functors

How do these relate?

$$\begin{tikzcd}
 & \mathrm{Alg}^{dg} \arrow[dl, "fgt_L"'] \arrow[dd, "fgt"'] & \\
\mathrm{Lie}^{dg} \arrow[ur, bend left, "U"] \arrow[dr, "fgt"'] & & \mathrm{CAlg}^{dg} \arrow[dl, "fgt"'] \\
 & \mathrm{Vect}^{dg} \arrow[ul, bend right, "\mathcal{L}"'] \arrow[uu, bend right, "T"'] \arrow[ur, bend right, "Sym"'] &
\end{tikzcd}$$

fgt = "forget"

There is a "not so forgetful" functor

$$\mathrm{Alg}^{dg} \xrightarrow{fgt_L} \mathrm{Lie}^{dg}$$

$$(A_*, \cdot) \longmapsto (A, [\,])$$

$$[a,b] = a\cdot b - (-1)^{ab}\, b\cdot a$$

There is $\overset{\text{tensor algebra functor}}{\mathcal{T}}$: $\mathrm{Vect}^{dg} \longrightarrow \mathrm{Alg}^{dg}$

$$V_* \longmapsto \mathcal{T}(V_*) = \bigoplus_n (V_*)^{\otimes n}$$

multiplication = concatenation

There is the free Lie algebra functor

$$\mathcal{f} : \mathrm{Vect}^{dg} \longrightarrow \mathrm{Lie}^{dg}$$

where $\mathcal{f}(V_*)$ is the sub dg Lie algebra in $\mathrm{fgt}_L(\mathcal{T}(V_*))$ generated by $V_* \cong V_*^{\otimes 1} \hookrightarrow T(V_*)$.

There is the universal enveloping algebra functor

$$U : \mathrm{Lie}^{dg} \longrightarrow \mathrm{Alg}^{dg}$$

where

$$U(g_*) = T(g_*) \big/ J_g$$

with J_g is the two-sided ideal generated by $\{ x \otimes y - (-1)^{xy} y \otimes x - [x,y] : x, y \in g \}$.

The objects $\mathcal{f}(V_*)$, $T(V_*)$, $U(g_*)$ satisfy universal properties (as they come from left adjoints)

E/Sti

Ex

$\mathfrak{g}_* \in \mathrm{Lie}^{dg}$, $A \in \mathrm{Alg}^{dg}$

given $\varphi : \mathfrak{g}_* \longrightarrow \mathrm{fgt}_L(A_*)$ a map of dg Lie algs,

there is a <u>unique</u> map of assoc dg algs

$$\tilde{\varphi} : U\mathfrak{g}_* \longrightarrow A_*$$

such that

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\varphi"] \arrow[dr] & \circlearrowright & A \\ & U\mathfrak{g}_* \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

In short, we have

$$\underline{T} \dashv \mathrm{fgt}, \quad \underline{F} \dashv \mathrm{fgt}, \quad \underline{U} \dashv \mathrm{fgt}_L$$

left adjoints

More about $U(\mathfrak{g}_*)$

There is a natural filtration

$$\begin{array}{ccccccccc} T(\mathfrak{g}) : & \underset{\substack{\| \\ k}}{\mathfrak{g}^{\otimes 0}} & \subseteq & \underset{k \oplus \mathfrak{g}}{\mathfrak{g}^{\otimes 0} \oplus \mathfrak{g}^{\otimes 1}} & \subseteq \cdots \subseteq & \bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i} & \subseteq \cdots \\ \downarrow & & & & & & \\ U(\mathfrak{g}) & k & \subseteq & '' & \subseteq \cdots \subseteq & (U\mathfrak{g})^{\leq n} & \subseteq \cdots \end{array}$$

Let $\mathrm{gr}(U\mathfrak{g}) = \bigoplus_{n=0}^{\infty} (U\mathfrak{g})^{\leq n} / (U\mathfrak{g})^{\leq n-1}$

It is a commutative dg algebra.

Then composite

$$\mathfrak{g}_* \hookrightarrow U^{\leq 1}(\mathfrak{g}) \longrightarrow \mathrm{gr}(U\mathfrak{g})$$

induces a map of dg commutative algebras

$$\psi_{\mathfrak{g}_*} : \mathrm{Sym}(\mathfrak{g}_*) \longrightarrow \mathrm{gr}(U\mathfrak{g}_*)$$

<u>Thm</u> (Poincaré-Birkhoff-Witt)

$\psi_{\mathfrak{g}_*}$ is an isomorphism.

<u>Pf</u> Tedious: pick a basis. $\square$

2 <u>Upgrade to Quillen adjunctions</u>

Equip Vect^{dg} with a model category structure where

$$W = \{\text{quasi-isomorphisms}\}$$

$$F = \{ f_* : M_* \to N_* \; : \; f_n \text{ surjective } \forall n \}$$

$$\mathrm{Cof} = \{ \text{degreewise injective} \}$$

We can transfer this model structure to Lie^{dg}, Alg^{dg}, and CAlg^{dg} via our adjunctions ($\mathcal{F}$, T, and Sym, respectively).

For example,

$$W_{\mathrm{Lie}} = W \cap \mathrm{Mor}_{\mathrm{Lie}} = \mathrm{fgt}^{-1}(W)$$

$$F_{\mathrm{Lie}} = F \cap \mathrm{Mor}_{\mathrm{Lie}} = \mathrm{fgt}^{-1}(F)$$

This generates a cofibrantly-generated model category on Lie^{dg}.

Rmk: On CAlg^{dg}, the analogous structure lets you begin DAG in characteristic zero. //

Thus $(\mathcal{F}, \mathrm{fgt})$, (T, fgt), $(\mathrm{Sym}, \mathrm{fgt})$ are Quillen adjunctions!

In fact, (U, fgt_L) is a Quillen adjunction too.

~~Lemma~~

Lemma

If $\phi : \mathfrak{g}_* \longrightarrow \mathfrak{g}'_*$ is a dg Lie algebra map,

then ϕ is a quasi-iso $\Leftrightarrow U\phi$ is a quasi-iso.

Proof

ϕ is a quasi-iso $\overset{\text{char zero}}{\Longleftrightarrow}$ $\mathrm{Sym}(\phi)$ is a quasi-iso

$\overset{\text{PBW}}{\Longleftrightarrow}$ $U(\phi)$ is a quasi-iso $\square$

Thus we have a triangle of Quillen adjunctions

$$\begin{tikzcd}
\mathrm{Lie}^{dg} \arrow[rr, bend left, "U"] \arrow[dr] & & \mathrm{Alg}^{dg} \arrow[ll] \arrow[dl] \\
& \mathrm{Vect}^{dg} \arrow[ul, bend left, "f"] \arrow[ur, bend right, "T"'] &
\end{tikzcd}$$

③ Lie algebra (co)homology

~~A~~ is equivalent to equip M_* with

a left $U(\mathfrak{g}_*)$-module structure

and a left $\mathfrak{g}_*$-representation structure.

Reminder:

invariant elements $= M^{\mathfrak{g}} = \{ m \in M : \mathfrak{g} \cdot m = 0 \}$

Coinvariants $= M_{\mathfrak{g}} = M / \mathfrak{g}M$

Moreover

trivial $\mathfrak{g}$-module

$$M^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(k, M)$$

$$M_{\mathfrak{g}} = k \otimes_{U\mathfrak{g}} M.$$

These are parts of adjunctions

$$\begin{tikzcd} \mathfrak{g}\text{-}\mathrm{Mod}^{dg} \arrow[r, bend left=50, "(\ )_{\mathfrak{g}}"] \arrow[r, bend right=50, "(\ )^{\mathfrak{g}}"'] & \mathrm{Vect}^{dg} \arrow[l, "\mathrm{triv}"' description] \end{tikzcd}$$

$\mathrm{triv}(M)$
''
trivial $\mathfrak{g}$-module structure

Hence $(-)^{\mathfrak{g}}$ = left exact

$(-)_{\mathfrak{g}}$ = right exact

<u>Def</u> The <u>Lie algebra homology of $\mathfrak{g}$</u> is the left derived functor

$$L_*(-_{\mathfrak{g}}) =: H_*(\mathfrak{g}, -)$$

and the <u>Lie algebra cohomology of $\mathfrak{g}$</u> is the right derived functor

$$R_*(-^{\mathfrak{g}}) =: H^*(\mathfrak{g}, -)$$

To calculate (co)homology, find a projective resolution of k as a Ug-module.

Idea: find an acyclic dg La containing g_* then take U of it to obtain a resolution of k

Construction

Define the "cone" of g, $Cn(g)_*$ by

$$Cn(g)_n = g_n \oplus \varepsilon g_{n-1}$$

formal notation

with differential

diff in g

$$D(x+\varepsilon y) = dx + y - \varepsilon dy$$

and bracket

$(-1)^x$

$$[x+\varepsilon y, x'+\varepsilon y'] = [x,x'] + \varepsilon\left([x,y'] + [y,x']\right)$$

Remark $Cn(g)_*$ is acyclic &

$$g_* \hookrightarrow Cn(g)_*$$

$$x \longmapsto x + \varepsilon 0$$

is an inclusion

Def The Chevalley-Eilenberg complex is

$$C_*(\mathfrak{g}, M) := U(\mathrm{Cn}(\mathfrak{g})_*) \underset{U\mathfrak{g}}{\otimes} M$$

Rmk $\varepsilon\mathfrak{g}[1] \hookrightarrow \mathrm{Cn}(\mathfrak{g})_*$ is an abelian subalgebra so $U(\varepsilon\mathfrak{g}[1]) = \mathrm{Sym}(\mathfrak{g}[1])$

By PBW, ~~the underlying graded vector space~~

$$U(\mathrm{Cn}(\mathfrak{g})_*) \underset{\xi}{\cong} \mathrm{Sym}(\mathfrak{g}[1]) \underset{k}{\otimes} U\mathfrak{g}$$

as $U\mathfrak{g}$-modules

$$\Rightarrow \quad C_*(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g}[1])$$

as graded vector spaces

$$C_*(\mathfrak{g}) = C_*(\mathfrak{g}, k)$$

What is the differential?

It is given by

$$D(x_1, \dots, x_n) = \sum_i (-1)^{x_1 \cdots x_{i-1}} x_1 \cdots x_{i-1} (dx_i) x_{i+1} \cdots x_n$$

$$+ \sum (-1)^{\sim} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n$$

$C_*(\mathfrak{g})$ has a natural filtration by

$$C_*(\mathfrak{g})^{\leq n} = \bigoplus_{i=0}^{n} \mathrm{Sym}^i(\mathfrak{g}[1])$$

so

$$\mathrm{gr}\, C_*(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g}_*)$$

as dg vector spaces

By definition,

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$$

Moreover

$$H^1(\mathfrak{g}, M) = \mathrm{Der}(\mathfrak{g}, M)$$

//

k is a "universal" Lie module:

every Lie algebra has the trivial rep

$$\Rightarrow \quad \mathfrak{g}_* \xrightarrow{\varphi} \mathfrak{g}_*' \rightsquigarrow C_*(\mathfrak{g}_*) \xrightarrow{C_*(\varphi)} C_*(\mathfrak{g}_*')$$

That is, we have

$$C_* : \mathrm{Lie}^{dg} \longrightarrow \mathrm{coCAlg}^{dg}$$

Take $C^*\mathfrak{g} := (C_*\mathfrak{g})^\vee = \mathrm{Hom}_k(C_*\mathfrak{g}, k)$

Then $C^*: \mathrm{Lie}^{dg} \longrightarrow (\mathrm{coCAlg}^{*s})^{op}$

DAG X, 2.2.6

Both C_*, C^* preserve quasi-isos &

hence have adjoints at the level

of homotopy categories

DAG X, 2.3.2

These do not admit right adjoints &

hence no Quillen adjoints